since for $\varkappa \leqslant {}^3/_4\,\pi$ each multiplier is positive, while for $\varkappa \geqslant {}^3/_4\,\pi$ with allowance for the increase of $c_2(\varkappa)$ we have

$$8J > (c_2 + 1)\ \ (\varkappa - {}^3/_4) - ({}^7/_2\,\varkappa + {}^1/_8) = f(\varkappa) \geqslant f({}^3/_4\ \ \pi) > 0$$

For $J \neq 0$ and $\lambda > 0$ Eq.(1.4) can have one stable stationary solution Q or two stable (with a considerable and a small $|Q|$) and one unstable.

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ON THE DETERMINATION OF THE SHAPE OF BODIES FORMED BY SOLIDIFICATION OF THE FLUID PHASE OF THE STREAM

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The problem of determination of the shape of bodies that have solidified in a moving fluid, of heat exchange between these and the fluid is encountered in domains such as underground construction by freezing water-saturated rocks, heat

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exchange in underground liquefied gas reservoirs, heat exchangers with molten metals, etc. An approximate representation of external temperature fields is investigated in the present work. This approximation is used for the formulation of fairly general methods for computing the shape of solid bodies formed during solidification of the stream liquid phase.

1. Statement of the problem. The stationary problem of determination of the shape of ice-type solid bodies that build up around cooling devices in a stream of perfect fluid is considered. The problem of freezing rock formations under filtration conditions is identically stated. The solid bodies formed in this manner are called equiponderant; they satisfy the condition of continuity of the heat flux through the boundary.

The relation between thermo- and hydrodynamics in the stream outside the body is important for the subsequent analysis. The thermal diffusivity equation in the moving fluid outside the body, after the Boussinesq transformation [1], is of the form

$$a^2\left(\frac{\partial^2 T}{\partial \varphi^2}+\frac{\partial^2 T}{\partial \psi^2}\right)=\frac{\partial T}{\partial \varphi} \qquad (1.1)$$

where a^2 is the thermal diffusivity of the fluid, $T\ (\varphi, \psi)$ is the external temperature field, and χ and φ are, respectively, the hydrodynamic stream function and the potential ($v = \operatorname{grad} \varphi$ is the velocity field). Terms in the left-hand part represent convective heat transfer along and across the fluid streamlines, respectively, and the right-hand part defines the convective heat transfer along streamlines.

It is clear that at high Péclet numbers $P = v_\infty l / a^2$ (v_∞ and l are the characteristic stream velocity and dimension of the body) along streamlines the convective heat transfer predominates and it is possible to neglect in Eq. (1.1) the term $T_{\varphi\varphi}$. This was noted in [2], where the scope of applicability of the abridged equation

$$a^2\frac{\partial^2 T}{\partial \psi^2}=\frac{\partial T}{\partial \varphi} \qquad (1.2)$$

was partly investigated by comparing it with the analytical solution of complete equation, which is known for the second boundary value problem.

The problem of lengthwise flow past a semi-infinite plate at constant temperature, defined by Eq. (1.1) was solved analytically in [3]. An unexpected result of that comparison was the exact agreement between the heat fluxes on the plate calculated by Eq. (1.1) and by the abridged equation (1.2) whose solution in this case is

$$T(\varphi, \psi) = T_\infty \operatorname{erf}\frac{\psi}{2a\sqrt{\varphi}} \qquad (1.3)$$

(T_∞ is the stream temperature away from the body, and the temperature of the body boundary is zero). This result justifies the use of the proposed method for flows past semi-infinite bodies (their boundary in coordinates φ, ψ is the semiaxis $\psi = 0$ and $\varphi \geqslant 0$).

The applicability of (1.3) to finite bodies requires further investigation. This equation is also of interest for the determination of the region of the thermal boundary layer. For this Eq. (1.1) was solved numerically on a computer. In dimensionless coordinates $\Phi = \varphi / \Delta\varphi$, $\Psi = \psi / \Delta\varphi$ (where $\Delta\varphi$ is the increment of φ between the stagnation points of the flow), the body boundary is represented on the Φ-axis by segment $0 \leqslant \Phi \leqslant 1$. The dimensionless temperature $u\ (\Phi, \Psi)$ is zero at the body boundary and unity at infinity. The numerical solution was obtained for various numbers $P = \Delta\varphi / a^2$ (we recall that $\Delta\varphi \sim v_\infty l$).

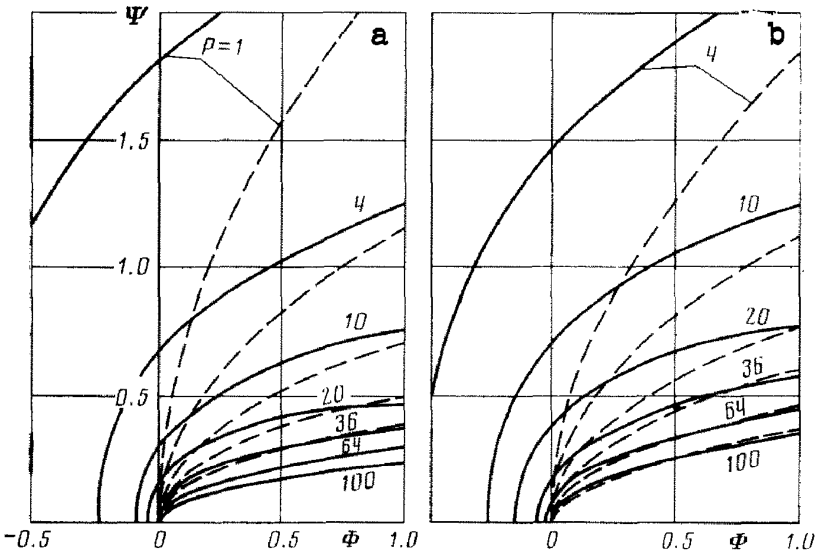

Fig. 1

Results of numerical solutions are shown in Fig. 1 by solid lines (isotherms $u = 0.01$ and $u = 0.1$ are shown, respectively, in Fig. 1, a and 1, b). For comparison the related isotherms determined by the approximate solution (1.3) are shown by dash lines. It will be seen that for $P \gg 50$ the exact and the approximate solutions are virtually the same.

Formula

$$g(\Phi) = -\frac{1}{2}\sqrt{\frac{\pi}{P}}\frac{1}{\sqrt{\Phi}}\int_0^\Phi \frac{\partial u}{\partial \Psi}\bigg|_{\Psi=0} d\Phi \tag{1.4}$$

where integration is carried out over the results of the numerical solution, represents the ratio of over-all heat fluxes to the body boundary obtained by the numerical solution to that calculated by (1.3) between the leading stagnation point and current value of Φ. Curves $g(\Phi)$ are represented in Fig. 2. These show that the total heat influx to the body can be calculated by the approximate solution already for $P \gg 10$ (with an error not exceeding 2%).

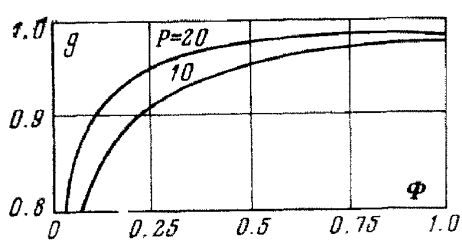

Fig. 2

The problem of determining the shape of the body with the use of solution (1.3) is considered below.

2. The shape of a finite body. We denote by Γ the contour of the solid body and by Γ_0 that of the cooling equipment. Contours Γ_0 and Γ are assumed to be symmetric about some straight line parallel to the flow of fluid. The physical plane $z = x + iy$ is oriented so that the

Ox-axis coincides with the above straight line and the coordinate origin is at the downstream edge of the body (Fig. 3, a). The complex flow potential $w(z) = \varphi(x, y) + i\psi(x, y)$ satisfies boundary conditions

$$\psi = 0, \quad z \in \Gamma, \quad -\infty < x \leqslant x_A, \; 0 \leqslant x < \infty \tag{2.1}$$

$$\overline{\frac{dw_1}{dz}} \to v_\infty, \quad z \to \infty; \quad w(z_A) = 0$$

(A is the leading stagnation point of the body and v_∞ is the velocity of flow at infinity).

The temperature distribution $T_1(x, y)$ inside the body is defined by the real part of the complex heat potential $w_1(z)$, i.e. $T_1 = \operatorname{Re} w_1(z)$.

Temperature readings are taken from the melting point, i.e. from the temperature at the body boundary

$$\operatorname{Re} w_1(z) = 0, \quad z \in \Gamma \tag{2.2}$$

Functions $w(z)$ and $w_1(z)$ and contour Γ are the unknowns. The condition of heat flux continuity at transition through the contour Γ is of the form

$$k_1 \frac{\partial T_1}{\partial n} = k \frac{\partial T}{\partial n} \tag{2.3}$$

where k and k_1 are thermal conductivities of the fluid and solid phases, respectively, and $\mathbf{n}$ is a normal to Γ.

Taking into account (2.1)—(2.3), (1.3), and that at the boundary $T = T_1 = 0$, we obtain [4, 5]

$$\left|\frac{dw_1}{dz}\right| = \lambda \left|\frac{d\sqrt{w}}{dz}\right|, \quad z \in \Gamma; \quad \lambda = \frac{2T_\infty k}{k_1 a \sqrt{f_1}} \tag{2.4}$$

We introduce the new complex variable $\zeta = \sqrt{z / l_0}$ (where l_0 is a characteristic dimension of contour Γ_0) and function $\omega(\zeta) = i\lambda\sqrt{w}$ (henceforth ζ is taken in the

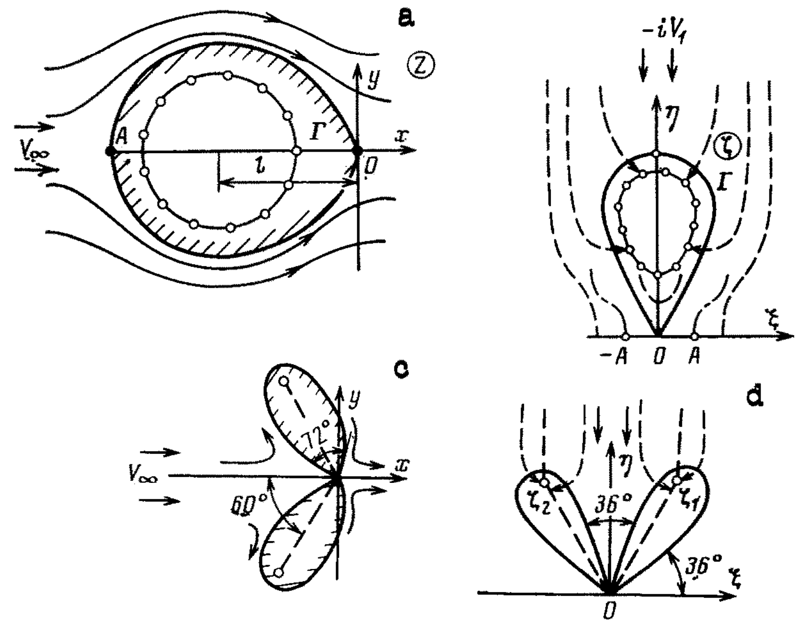

Fig. 3

upper half-plane, see Fig. 3, b). Function $\omega(\zeta)$ is analytic in the upper half-plane outside the transformed contour Γ, and $w_1(\zeta) = w_1[\omega(\zeta)]$ is an analytic function inside contour Γ and outside Γ_0. Conditions (2.1), (2.2) and (2.4) become

$$\overline{\frac{d\omega}{d\zeta}} \to -iv_1, \quad \zeta \to \infty; \quad v_1 = \lambda\sqrt{l_0 v_\infty} \tag{2.5}$$

$$\operatorname{Im}\omega(\zeta) = 0, \quad \zeta = 0, \quad \eta \geqslant \eta^*; \quad \eta^* = \sqrt{|x_A|/l_0} \tag{2.6}$$

$$\operatorname{Re} w_1(\zeta) = \operatorname{Re}\omega(\zeta) = 0, \quad \zeta \in \Gamma, \quad \eta = 0 \tag{2.7}$$

$$|w_1'(\zeta)| = |\omega'(\zeta)|, \quad \zeta \in \Gamma \quad (\zeta = \xi + i\eta) \tag{2.8}$$

Conditions (2.7) and (2.8) show that functions $w_1(\zeta)$ and $\omega(\zeta)$ are analytic continuations of each other through contour Γ.

The problem has been thus reduced to the determination of a function that is analytic in the upper half-plane and outside Γ_0, and which satisfies conditions (2.5) and (2.7), and the boundary condition at the known transformed contour Γ_0 (condition (2.6) follows from the symmetry (of Γ_0) and is included for clarity).

As an example, let us consider a solid body formed by the freezing of fluid around a circular bank of n uniformly spaced point cold sources of capacity q each. The sources are located at points z_j on a circle of radius R_0 and center at point $(-l, 0)$ (see Fig. 3, a). The conditions at the bank Γ_0 amount to that the function $\omega(\zeta)$ must have logarithmic singularities of the same capacity at points ζ_j.

In the considered case functions $\omega(\zeta)$ can be treated on the basis of (2.5) — (2.7) as the complex potential of the fictitious stream which flows at velocity v_1 toward the sources of capacity q, located at points ζ_j, along the imaginary axis. The real axis ξ must be an isoline of the zero potential $\operatorname{Re}\omega = 0$.

The explicit expression for the potential is obtained by continuing the flow into the lower half-plane and locating sources of opposite capacity $-q$ at points $\bar{\zeta}_j$

$$\omega(\zeta) = \frac{q}{2\pi}\left(\beta i\zeta - \sum_j \ln\frac{\zeta - \zeta_j}{\zeta - \bar{\zeta}_j}\right), \quad \beta = \frac{2\pi v_1}{q} \tag{2.9}$$

The potentials in the physical plane z are derived from this by formulas

$$w_2(z) = \omega(\sqrt{z/R_0}), \quad w(z) = \frac{1}{\beta^2}\omega^2(\sqrt{z/R_0})$$

So far these functions are not entirely determined, since the distance l which defines the position of the coordinate origin in the z-plane is not known.

It is convenient to carry out further investigation in the ζ-plane. Formulation of the problem requires the contour Γ to be represented by a single closed curve that has a unique common point with the ξ-axis, namely $\zeta = 0$. The condition that the curve which represents Γ must emanate from point $\zeta = 0$, means that the equation $\operatorname{Re}\omega = 0$ must have at that point a root of higher than the first multiplicity. This requirement leads to the following formula:

$$\beta = 2\sum_j \frac{\eta_j}{\xi_j^2 + \eta_j^2} \quad (\zeta_j = \xi_j + i\eta_j) \tag{2.10}$$

The parameter l is determined from this, if the formulas

$$\zeta_j = \left[\exp\left(i\frac{2\pi}{n}j\right) - \frac{l}{R_0}\right]^{1/2}, \quad \beta = \frac{4T_\infty k}{qa}\sqrt{\pi v_\infty R_0} \tag{2.11}$$

which follow from (2.4), (2.5) and (2.9), are taken into account. Equation for the solid body contour Γ is determined by the condition $\operatorname{Re} \omega(z) = 0$. Specific examples can be found in [6].

Substituting q/n for q and assuming $R_0 \to 0$, we obtain one source of capacity q. The distance l from the source to the coordinate origin is [5]

$$l = \frac{q^2 a^2}{4\pi T_\infty^2 k^2 v_\infty} \qquad (2.12)$$

Note that in this problem the thermal conductivity of the body does not appear in the final formulas. This is as it should be in problems with sources, since the heat potential can be introduced for $k_1 T_1$ and not for T_1.

The considered method of problem solution is unsuitable for a multiply connected contour Γ. However, in practice it is the determination of the order of connectedness of contour Γ that is the most interesting. In some cases the linking of bodies formed around individual cold sources is the main requirement, while in others it is the avoidance of such linking. This problem can be readily investigated in the case of a small number of sources.

Let us consider, as an example, two cold sources located on a line perpendicular to the stream (Fig. 3, c). Condition (2.10) is assumed to be satisfied, i.e. the branches of contour Γ originate at point O. For small β the region of negative potentials contains in the ζ-plane both sources, and contour Γ is represented by a single curve. With increasing β the division of Γ into two branches occurs at some instant. This means that in the neighborhood of point $\zeta = 0$ the terms containing ζ^2 and ζ^3 in the expansion of $\omega(\zeta)$ must vanish.

Computation in [6] yielded the limit value $\beta^* = 2^{1/2} \cdot 3^{3/4}$, from which, with allowance for formula (2.11), we obtain the following condition for simple-connectedness of contour Γ

$$\frac{2\sqrt{2\pi}}{3^{3/4}} \frac{T_\infty k}{qa} \sqrt{v_\infty R_0} < 1$$

There is no solution of the boundary value problem for a multiply-connected contour Γ. Because of this the approximate method [4] is used. It consists of substituting a suitable curve for the unknown contour Γ and satisfying the equality of heat fluxes in the mean.

Let us find the result of the approximate solution in the case of a body solidified around a single point cold source . In [5] the shape of the body boundary was determined with observance of the strict condition (2.3). The length of the body was found to be equal $1.44\ l$ and its transverse dimension equal $1.04\ l$, where l is calculated by formula (2.12). Assuming the contour of the body to be approximately a circle of radius R and using (2.4), we obtain

$$q = \frac{4T_\infty k}{a\sqrt{\pi}} \Delta \sqrt{w}$$

where $\Delta\sqrt{w}$ is the increment of $\sqrt{w}$ between stagnation points. The flow is determined by the Joukovski potential. As the result, for the radius of the body we obtain the formula

$$R = \frac{\pi q^2 a^2}{64 T_\infty^2 k^2 v_\infty}$$

Comparison with the exact solution, in which $D = 1.24\ l$ is taken as the mean dimension of the body, yields the ratio $2R/D = 0.995$.

3. Solid bodies formed on semi-infinite walls in a lengthwise stream. Profile Γ_0 of the cooling wall at temperature $-T_0$ is represented by two (generally nonsymmetric) branches of the curve, which are directed along the positive x-axis. We assume that at infinity the transverse dimensions of contour Γ_0 do not increase faster than $\sqrt{x}$. As before, the hydrodynamic potential $w(z)$ is chosen so that at the leading stagnation point A of contour Γ of the solidified body $w_A = 0$. Let the contour Γ_0 be mapped by function $z = f(\zeta)$ onto the upper half-plane of the variable $\zeta = \xi + i\eta$ with the correspondence of infinitely distant points. On these assumptions about contour Γ_0 function $f(\zeta)$ is at infinity of order ζ^2. For function $\omega(\zeta) = \sqrt{w}$ and the heat potential $w_1(\zeta)$ we have conditions (2.2) and (2.4), as well as the following condition:

$$\operatorname{Im}\omega(\zeta) = 0, \qquad \zeta \in \Gamma; \qquad \operatorname{Re} w_1 = -T_0, \quad \eta = 0 \tag{3.1}$$

The last condition assumes the form

$$\frac{dw}{dz} = \frac{2\omega}{f'(\zeta)}\frac{dw}{d\zeta} \to v_\infty, \quad \zeta \to \infty \tag{3.2}$$

Let us prove that contour Γ is represented in the ζ-plane by the straight line $\zeta = \xi + i\sqrt{a_0}$ parallel to the real axis, and function w_1 and ω are of the form

$$w_1(\zeta) = -\frac{T_0}{\sqrt{a_0}}(i\zeta + \sqrt{a_0}), \quad \omega(\zeta) = \frac{T_0}{\lambda\sqrt{a_0}}(\zeta - i\sqrt{a_0}) \tag{3.3}$$

A direct test shows that conditions (2.2), (2.4) and (3.1) are satisfied. Condition (3.2) can be satisfied since $\omega(\zeta)$ is linear and $f'(\zeta)$ is at infinity of order ζ. This condition associates $\sqrt{a_0}$ with v_∞ and with parameters of transformation of $f(\zeta)$, and we then obtain the equation of the contour in a parametric form. Thus for solving this problem it is sufficient to determine the function that maps the form of the known contour Γ_0 onto the upper half-plane.

Let us consider some examples. Let Γ_0 be the parabola $y^2 = 4b(x+b)$ which is mapped onto the upper half-plane by function $z = f(\zeta) = (\zeta + i\sqrt{b})^2$. Substituting the expression for $\omega(\zeta)$ from (3.3) into (3.2), we obtain for a_0 the formula

$$a_0 = \frac{\pi T_0^2 k_1^2 a^2}{4T_\infty^2 k^2 v_\infty} \tag{3.4}$$

The contour Γ is also represented by a parabola

$$y^2 = 4(\sqrt{a_0} + \sqrt{b})^2[x + (\sqrt{a_0} + \sqrt{b})^2] \tag{3.5}$$

which is confocal to Γ_0. Thickness of the solidified body frontal part is $a_0 + 2\sqrt{a_0 b}$. The particular case when Γ_0 is a semi-infinite wall $x \geqslant 0$ obtains for $b = 0$. Formula (3.4) for a_0 remains valid, and the potentials become

$$w_1 = -\frac{T_0}{\sqrt{a_0}}(i\sqrt{z} + \sqrt{a_0}), \quad w = v_\infty(\sqrt{z} - i\sqrt{a_0})^2 \tag{3.6}$$

Let us determine the heat influx $Q(x)$ to the wall over the length x. Allowing for (3.4) and (3.6) we obtain

$$Q(x) = 2k_1\int_0^x \frac{\partial T_1}{\partial n}\bigg|_{y=0} dx = 2k_1\int_0^x \left|\frac{dw_1}{dz}\right|_{z=x} dx = \frac{4}{\sqrt{\pi}}\frac{T_\infty k\sqrt{v_\infty}}{a}\sqrt{x}$$

Note that the above formula contains only parameters of external stream, which means that for equal external conditions the heat exchange is the same for any body. The internal parameters T_0 and k_1 determine the size of such bodies.

If the contour Γ_0 has the form of a half-band $2h$ wide (edges of the half-band are at points $z = \pm ih$), then the mapping function is of the form

$$z = f(\zeta) = -i\frac{2h}{\pi}(\zeta\sqrt{1-\zeta^2} + \arcsin\zeta)$$

Condition (3.2) implies that in this case

$$a_0 = \frac{\pi^2 T_0^2 k_1^2 a^2}{8T_\infty^2 hk^2 v_\infty}$$

This dimensionless quantity determines the shape of contour Γ, whose parametric equation is of the form

$$x + iy = -i\,\frac{2h}{\pi}\,[(\xi - i\sqrt{a_0})\sqrt{1-(\xi+i\sqrt{a_0})^2} + \arcsin(\xi + i\sqrt{a_0})]$$

The thickness of the body at $\xi = 0$ is

$$AO = \frac{2h}{\pi}\,[\sqrt{a_0(a_0+1)} - \ln(\sqrt{a_0+1} - \sqrt{a_0})]$$

The proposed method can be found useful for determining the frontal and side parts of finite bodies, if the finite cooling equipment is extended downstream by, for example, a semi-infinite wall. Because of the physical aspects of the statement of the problem considered here, thermal perturbations from the body aft-part virtually do not extend upstream. Of course the aft-part affects the frontal one but that effect diminishes with increasing Péclet number.

Below we present the results of computations for a cooling pipe of radius R_0 (other variants of solution are given in [4, 5]). The contour Γ_0 is assumed to be a circle of radius R_0 with its center at the coordinate origin and the emerging from it part of semi-axis $x > 0$. We have

$$z = f(\zeta) = \frac{R_0}{4}(\zeta + \sqrt{\zeta^2 - 4})^2$$

$$x + iy = \frac{R_0}{4}\,[\xi + i\sqrt{a_0} + \sqrt{(\xi + i\sqrt{a_0})^2 - 4}\,]^2 \quad (x, y \in \Gamma)$$

$$d = \frac{R_0}{2}\,[a_0 + \sqrt{a_0(a_0+4)}], \qquad Q = \frac{8}{\sqrt{\pi}}\,\frac{T_\infty^2 k\sqrt{R_0 v_\infty}}{a}$$

$$a_0 = \pi T_0^2 k_1^2 a^2 / (4T_\infty^2 R_0 k^2 v_\infty)$$

where d is the thickness of the body leading edge and Q is the thermal flux per unit length of the pipe.

4. Three-dimensional axisymmetric problems. There are no exact solutions for three-dimensional problems. However it can be expected on the basis of solutions of plane problems that with a successful selection of approximation of the unknown shape of the body and by satisfying the boundary condition in the mean, a result close to the true will be obtained.

For an axisymmetric flow we shall use the following orthogonal curvilinear coordinates q_1, q_2 and q_3 : $q_1 = \varphi(x, y, z)$ which represents the flow potential ($v = \operatorname{grad}\varphi$), $q_2 = \psi(x, y, z)$ which is the streamfunction such that the discharge through the ring layer is equal $2\pi\Delta\psi$, and q_3 is a circular coordinate with respect to the coordinate axis of symmetry. For the single velocity component v_φ along the streamline we have [7] $v_\varphi = 1 / H_1 = 1 / (H_2 H_3)$, where H_1, H_2 and $H_3 = \rho$ are Lamé coefficients, with ρ denoting the distance from the axis of symmetry. In a stream of fluid the temperature diffusivity equation reduces to

$$a^2\left[\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial}{\partial \psi}\left(\rho^2 \frac{\partial T}{\partial \psi}\right)\right] = \frac{\partial T}{\partial \varphi} \quad (h_1 = \rho h_2) \tag{4.1}$$

For high Péclet numbers P we substitute in (4.1) for function $\rho^2(\varphi, \psi)$ its value at the body boundary $\rho^2(\varphi, 0)$ and, as previously, omit the term $\partial^2 T / \partial \varphi^2$.

For the considered problem the solution of the obtained equation is of the form

$$T(\varphi, \psi) = T_\infty \operatorname{erf} \frac{\varphi}{2\sqrt{\chi(\varphi)}}, \quad x(\varphi) = a^2 \int_{\varphi_A}^{\varphi} \rho(\varphi, 0)\, d\varphi \tag{4.2}$$

Because of the known properties of solutions of Eq.(4.1) without the term $\partial^2 T / \partial \varphi^2$ it is not unreasonable to expect that the substitution of $\rho(\varphi, 0)$ for $\rho(\varphi, \psi)$ will not result in a very great difference in solutions. This assumption was tested in [8] where exact solution of the complete equation (4.1) was obtained by numerical methods for the case of flow past a sphere for $P = 180$. The comparison of isolines and heat fluxes obtained by the exact numerical solution and by the approximate analytical solution (4.2) shows that these are virtually the same (within the accuracy of the numerical solution).

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